On the uniqueness of the surface sources of evoked potentials

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Abstract.

The uniqueness of a surface density of sources localized inside a spatial region R and producing a given electric potential distribution in its boundary B_0 is revisited. The situation in which R is filled with various metallic subregions, each one having a definite constant value for the electric conductivity is considered. It is argued that the knowledge of the potential in all B_0 fully determines the surface density of sources over a wide class of surfaces supporting them. The class can be defined as a union of an arbitrary but finite number of open or closed surfaces. The only restriction upon them is that no one of the closed surfaces contains inside it another (nesting) of the closed or open surfaces.

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I. INTRODUCTION.

The uniqueness problem for the sources of the evoked potential in the brain is a relevant research question due to its role in the development of cerebral electric tomography [1], [2] , [3], [4] . Since long time ago, it is known that the general inverse problem of the determination of volumetric sources from the measurement of the potential at a surface is not solvable in general [5], [6]. However, under additional assumptions about the nature of the sources, solutions can be obtained [7], [8], [9]. The supplementary assumptions can be classified in two groups: the physically grounded ones, which are fixed by the nature of the physical problem and the ones which are imposed by invoking their mathematical property of determining a solution, but having in another hand, a weak physical foundation. The resumed situation implies that the determination of physical conditions implying the uniqueness of the sources for the evoked potentials remains being an important subject of study. Results in this direction could avoid the imposition of artificial conditions altering the real information on the sources to be measured.

The question to be considered in this work is the uniqueness of the sources for evoked potentials under the assumption that these sources are localized over surfaces. This issue was also treated in Ref. [1] by including also some specially defined volumetric sources. The concrete aim here is to present a derivation of the results enunciated in [1] for the case of open surfaces and to generalize it for a wider set of surfaces including closed ones.

We consider that the results enunciated in Ref. [1] are valid and useful ones. Even more, we think that a relevant merit of that paper is to call for the attention to the possibility for the uniqueness for classes of surface density of sources. Specifically, in our view, the conclusion stated there about the uniqueness of the sources of evoked potentials as restricted to sources distributed in open surfaces is effectively valid. In the present work, the central aim is to extend the result for a wider set of surfaces including closed ones by also furnishing an alternative way to derive the uniqueness result. The uniqueness problem for the special class of volumetric sources discussed in [1] is not considered here in any way.

The physical system under consideration is conformed by various volumetric regions, each of them having a constant value of the conductivity, separated by surface boundaries at which the continuity equations for the electric current is obeyed. It should pointed out that the special volumetric sources examined in Ref. [1] are not addressed here. The precise definition of the generators under examination is the following. The sources are assumed to be defined by continuous and smooth surface densities lying over a arbitrary but finite number of smooth open or closed surfaces. The unique constraint to be imposed on these surfaces is that there is no nesting among them. That is, there is no closed surface at which interior another open or closed of the surfaces resides. This class of supports expands the one considered in Ref. [1] and in our view is sufficiently general to create the expectative for the practical applications of the results. It should be stressed that the boundaries between the interior metallic regions are not restricted by the "non-nesting" condition. That is, the fact that the skull and the few boundaries between cerebral tissues can be visualized as nearly closed surface does not pose any limitation on the conclusion. The "non-nesting" condition should be valid only for the surfaces in which the sources can be expected to reside. For example, if by any mean we are sure that the sources stay at the cortex surface, then the uniqueness result apply whenever the portion of the cortex implied does not contains any closed surface.

The paper is organized as follows. An auxiliary property is derived in the form of a theorem in the Section II. In Section III the proof of uniqueness for the kind of sources defined above is presented.

II. GREEN THEOREM AND FIELD VANISHING CONDITIONS

Let us consider the potential ϕ generated by a source distribution concentrated in the "non-nested" set of open or closed surfaces defined in last Section, which at the same time are contained within a compact and simply connected spatial region R. The set R, as explained before, is formed by various connected subregions $R_i,\ i = 0, 1, ...n$ each of them filled with

a metal having a constant conductivity σ_i. Also, let B_{ij} the possibly but non necessarily existing, boundary between the subregions R_i and R_j and B_0 the boundary of R. For the sake of a physical picture, we can interpret B_0 as the surface of the skull, R as the interior of the head and the subregions R_i as the ones containing the various tissues within the brain. It is defined that the exterior space of the head corresponds to R_0. In addition, let $S_i,\ i = 1, ...m$ the surfaces pertaining to the arbitrary but finite set S of non-nested open or closed surfaces in which the sources are assumed to be localized. The above mentioned definitions are illustrated in Fig.1.

Then, the Poisson equation satisfied by the potential ϕ in the interior region of R can be written as

$$\nabla^2 \phi (\overrightarrow{x}) = \frac{g(\overrightarrow{x})}{\sigma(\overrightarrow{x})}, \tag{1}$$

$$g(\overrightarrow{x}) = -\overrightarrow{\nabla}.\overrightarrow{J}(\overrightarrow{x}), \tag{2}$$

where $\overrightarrow{J}$ are the impressed currents (for example, generated by the neuron firings within the brain) and the space dependent conductivity is defined by

$$\sigma(\overrightarrow{x}) = \sigma_i \quad for\ \overrightarrow{x} \in R_i. \tag{3}$$

It should be noticed that the conductivities are different from zero only for the internal regions to R. The vacuum outside is assumed to have zero conductivity and the field satisfying the Laplace equation. In addition outside the support of the sources where $g = 0$ the Laplace equation is also satisfied.

The usual boundary conditions within the static approximation, associated to the continuity of the electric current at the boundaries, take the form

$$\sigma_i \frac{\partial \phi}{\partial n_i} |_{x\ \in\ B_{ij}} = \sigma_j \frac{\partial \phi}{\partial n_j} |_{x\ \in\ B_{ij}}, \tag{4}$$

where ∂n_i symbolizes the directional derivative along a line normal to B_{ij} but taken in the limit of $x- > B_{ij}$ from the side of the region R_i.

A main property is employed in this work in obtaining the claimed result. In the form of a theorem for a more precise statement it is expressed as

Theorem.

Let ϕ is a solution of the Laplace equation within an open and connected spatial region R^{*}. Assume that φ have a vanishing electric field over an open section of certain smooth surface S^{*} which is contained in an open subset Q of R^{*}. Let the points of the boundaries between Q and R^{*} have a minimal but finite distance among them. Then, the potential ϕ is a constant over any open set contained in R^{*}.

As a first stage in the derivation of this property, let us write the Green Theorem as applied to the interior of the open region Q defined in the Theorem 1 in which a field φ satisfies the Laplace equation. Then, the Green Theorem expresses φ evaluated at a particular interior point $\overrightarrow{x}$ in terms of itself and its derivatives at the boundary B_Q as follows.

$$\varphi\left(\overrightarrow{x}\right)=\int_{B_Q} d\overrightarrow{s}\,.\left(\frac{1}{\left|\overrightarrow{x}-\overrightarrow{x}'\right|}\overrightarrow{\nabla_{x'}}\,\varphi\left(\overrightarrow{x}'\right)-\overrightarrow{\nabla_{x'}}\left(\frac{1}{\left|\overrightarrow{x}-\overrightarrow{x}'\right|}\right)\varphi\left(\overrightarrow{x}'\right)\right) \tag{5}$$

where the integral is running over the boundary surface B_Q which is described by the coordinates $\overrightarrow{x}'$. This relation expresses the potential as a sum of surface integrals of the continuous and bounded values of φ and its derivatives. Those quantities are in addition analytical in all the components of $\overrightarrow{x}$, if the point have a finite minimal distance to the points in B_Q. These properties follow because $Q\subset R^{*}$ and then, φ satisfies the Laplace equation in any open set in which Q and its boundary is included. But, due to the finite distance condition among the point $\overrightarrow{x}$ and the points of B_Q, the expression (5) for φ should be an analytical function of all the coordinates of $\overrightarrow{x}$. Figure 2 depicts the main elements in the formulation of the Green Theorem.

Further, let us consider that S^{*} is siting inside the region Q. Then, as this surface is an equipotential and also the electric field over it vanishes, it follows that no line of force can have a common point with it. This is so because the divergence of the electric field vanishes, then it is clear that the existence of nonvanishing value of the electric field at another point of the line of force will then contradicts the assumed vanishing of the divergence. Therefore, the lines of forces in any sufficiently small open neighborhood containing a section of S^{*}

should tend to be parallel to this surface on approaching it, or on another hand, the electric field should vanish. Next, it can be shown that in such neighborhoods the lines of forces can not tend to be parallel.

Let us suppose that lines of forces exist and tend to be tangent to the surface S^* and consider the integral form of the irrotational property of the electric field as

$$\oint_C \overrightarrow{E}.\, d\,\overrightarrow{l} = \int_{C_1} \overrightarrow{E}.\, d\,\overrightarrow{l} + \int_{C_2} \overrightarrow{E}.\, d\,\overrightarrow{l} = \int_{C_1} \overrightarrow{E}.\, d\,\overrightarrow{l} = 0 \tag{6}$$

where the closed curve C is constructed as follows: the piece C_1 coincides with a line of force, the piece C_2 is fixed to rest within the surface S^* and the other two pieces necessary to close the curve are selected as being normal to the assumed existing family of lines of forces. The definitions are illustrated in Fig. 3. By construction, the electric field is colinear with the tangent vector to C_1 and let us assume that we select the segment of curve C_1 for to have a sufficiently short but finite length in order that the cosine associated to the scalar product will have a definite sign in all C_1. This is always possible because the field determined by (5) should be continuous. Then Eq. (6) implies that the electric field vanish along all C_1 as a consequence of the integrand having a definite sign and then should vanish identically. Since this property is valid for any curve pertaining to a sufficiently small open interval containing any particular open section of S^*, it follows that in certain open set containing S^* there will be are no lines of forces, or what is the same, the electric field vanish.

To finish the proof of the theorem, it follows to show that if φ and the electric field vanish within a certain open neighborhood N, included in an arbitrary open set O pertaining to the region R^* in which the Laplace equation is obeyed, then φ and the electric field vanish in all O . Consider first that Q is an open set such that $O \subset Q$ and also suppose that the smallest distance form a point in O to the boundary B_Q of Q has the finite value δ. Then, the Green Theorem (5) as applied to the region Q expresses that the minimal radius of convergence of φ considered as analytical function of any of the coordinates is equal or greater than δ.

Imagine now a curve C starting in an interior point P of N and ending at any point P_1 of O. Assume that C is formed by straight lines pieces (See Fig. 4). It is then possible to define

φ as a function of the length of arc s of C as measured form the point P. It should be also valid that in any open segment of C, not including the intersection point of the straight lines, the potential φ is an analytical function of s. Furthermore, let consider C as partitioned in a finite number of segments of length $\sigma < \delta$. Suppose also, that the intersection points of the straight lines are the borders of some of the segments. It can be noticed that φ vanishes in any segment of C starting within N because it vanishes in N exactly. Thus, if φ and the electric field are not vanishing along all C, there should be a point over the curve in which the both quantities do not vanish for an open region satisfying $s > s_o$, and vanish exactly for another open interval obeying $s < s_o$. However, in this case, all the derivatives of φ of the electric field over s vanish at s_o. This property in addition with the fact that the Taylor series around s_o should have a finite radius of convergence $r > \delta$, as it assumed in the Theorem 1, leads to the fact that φ and the electric field should vanish also for $s > s_o$. Henceforth, the conclusion of the Theorem 1 follows: the potential φ and its corresponding electric field vanish at any interior point of R^*.

III. UNIQUENESS OF THE NON-NESTING SURFACE SOURCES

Let us argue now the uniqueness of the sources which are defined over a set of non nested surfaces S producing specific values of the evoked potential ϕ at the boundary B_0 of the region R. For this purpose it will be assumed that two different source distributions produce the same evoked potential over B_0. The electrostatic fields in all space associated to those sources should be different as functions defined in all space. They will be called ϕ_1 and ϕ_2. As usual in the treatment of uniqueness problems in the linear Laplace equation, consider the new solution defined by the difference $\phi = \phi_{1-}\phi_2$. Clearly φ corresponds to sources given by the difference of the ones associated to ϕ_1 and ϕ_2. It is also evident that ϕ has vanishing values at B_0. Then, since the sources are localized at the interior of R and ϕ satisfies the Laplace equation with zero boundary condition at B_0 and at the infinity, it follows that the field vanishes in all R_0, that is, in the free space outside the head. Therefore, it follows

that the potential and the electric field vanish in all B_0 when approaching this boundary from the free space (R_0). The continuity of the potential, the boundary conditions (3) and the irrotational character of the electric field allows to conclude that ϕ and the electric field also vanish at any point of B_0 but now when approaching it from any interior subregion R_i having a boundary B_{i0} with the free space. Moreover, if the boundary surface of any of these regions which are in contact with the boundary of R is assumed to be smooth, then it follows from Theorem 1 that the potential ϕ and its the electric field vanish in all the open subsets of R_i which points are connected through its boundary B_{i0} with free space by curves non-touching the surfaces of S. It is clear that this result hold for all the open subsets of these R_i in which Laplace equation is satisfied excluding those which are also residing inside one of the closed surfaces S_i in the set S.

It is useful for the following reasoning to remark that if we have any boundary B_{ij} between to regions R_i and R_j, and the potential ϕ and the electric field vanish in certain open (in the sense of the surface) and smooth regions of it, then Theorem 1 implies that the potential and the electric field also vanish in all the open subsets of R_i and R_j which are outside any of the closed surfaces in S. Since the sources stay at the surfaces in S the field ϕ in some open region of R included inside certain of the closed surfaces S_i will not necessarily satisfy the Laplace equation in any interior point of R and Theorem 1 is not applicable.

Let us consider in what follows a point P included in a definite open vicinity of a subregion R_i. Suppose also that P is outside any of the closed surfaces in S . Imagine a curve C which join P with the free space and does not touch any of the surfaces in S. It is clear that, if appropriately defined, C should intersect a finite number of boundaries B_{ij} including always a certain one B_{j0} with free space. Let us also assume that C is adjusted in a way that in each boundary it crosses, the intersection point is contained in a smooth and open vicinity (in the sense of the surface) of the boundary (See Fig. 1 and 5). Then, it also follows that the curve C can be included in open set O_C having no intersection with the non-nested surfaces in S. This is so because the region excluding the interior of the closed surfaces in S is also connected if the S_i are disjoint . But, from Theorem 1 it follows that ϕ and the electric field

must vanish in all O_C. This should be the outcome because the successive application of the Theorem 1 to the boundaries intersected by the curve C permits to recursively imply the vanishing of ϕ and the electric field in each of the intersections of O_C with the subregions R_i through which C passes. The first step in the recursion can be selected as the intersection of C with B_{j0} at a point which by assumption is contained in an open neighborhood of the boundary B_{j0}. As the electric field and ϕ vanish at free space, the fields in the first of the considered intersection of Oc should vanish. This fact permits to define another open and smooth neighborhood of the next boundary intersected by C in which the field vanish and so on up to the arrival to the intersection with the boundary of the region R_i containing the ending of C at the original point P. Therefore, the electric field and the potential should vanish at an arbitrary point P of R with only two restrictions: 1) P to be contained in an open neighborhood of some R_i and 2) P to reside outside any of the surfaces in S. Thus, it is concluded that the difference solution ϕ and its corresponding electric field, in all the space outside the region containing the sources vanish. Henceforth, it implies that the difference between the two source distributions also should be zero over any of the open surface in the set S. This is necessary because the flux going out from any small piece of the considered surface is zero, which means that the assumed continuous density of surface sources exactly vanish. This completes the proof of the conclusion of Ref. [1] in connection with sources supported by open surfaces. It only rests to show that the sources are also null over the closed S_i.

Before continuing with the proof, it is illustrative to exemplify from a physical point of view how the presence of nested surfaces among the S_i destroys the uniqueness. For this aim let us let us consider that a closed surface S_i has another open or closed of the surface S_j properly contained inside it. That means that an open set containing S_j is contained inside S_i. Imagine also that S_i is interpreted as the surface of a metal shell connected to the ground; that is, to a zero potential and that the surface S_j is the support of an arbitrary density of sources. As it is known from electrostatics theory, the charge density of a metal connected to the ground is always capable to create a surface density of charge at S_i such that it exactly

cancels the electric field and the potential at the outside of S_i, in spite of the high degree of arbitrariness of the charge densities at the interior. That is, for nested surfaces in S, it is not possible to conclude the uniqueness, because at the interior of a nesting surface, and distributed over the nested ones, arbitrary source distributions can exist which determine exactly the same evoked potential at the outside boundary B_0.

Let us finally show that if no nesting exists the uniqueness also follows. Consider any of the closed surfaces, let say S_i. As argued before ϕ and the electric field vanish at any exterior point of S_i pertaining to certain open set containing S_i. Then, the field created by the difference between the sources associated to the two different solutions assumed to exist should be different from zero only at the interior region. That zone, in the most general situation can be filled by a finite number of metallic bodies with different but constant conductivities. The necessary vanishing of the interior field follows from the exact conservation of the lines of forces for the ohmic electric current as expressed in integral form by

$$\int d\overrightarrow{s}.\,\sigma\left(\overrightarrow{x}\right)\overrightarrow{E}\left(\overrightarrow{x}\right)=0. \tag{7}$$

Let us consider a surface T defined by the all the lines of forces of the current vector passing through an arbitrarily small circumference c which sits on a plane being orthogonal to a particular line of force passing through its center. Let the center be a point at the surface S_i . Because, the above defined construction, all the flux of the current passing trough the piece of surface of S_i (which we will refer as p) intersected by T is exactly equal to the flux through any intersection of T with another surface determining in conjunction with p a closed region. By selecting a sufficiently small radius for the circumference c it can be noticed that the sign of the electric field component along the unit tangent vector to the central line of forces should be fixed. This is so because on the other hand there will be an accumulation of charge in some closed surface. Now, let us consider the fact that the electric field is irrotational and examine a line of force of the current density which must start at the surface S_i. It should end also at S_i, because in another hand the current density will not be divergence less. After using the irrotational condition for the electric field in the form

$$\oint_C \overrightarrow{E}.\, d\overrightarrow{l} = \int_{C_1} \overrightarrow{E}.\, d\overrightarrow{l} + \int_{C_2} \overrightarrow{E}.\, d\overrightarrow{l} = \int_{C_1} \overrightarrow{E}.\, d\overrightarrow{l} = 0 \tag{8}$$

in which C_1 is the line of force starting and ending at S_i and C_2 is a curve joining the mentioned points at S_i but with all its points lying outside S_i where $\phi = \phi_1$-ϕ_2 and the electric field vanish. Let us notice that the electric field and the current have always the same direction and sense as vectors, because the electric conductivity is a positive scalar. In addition, as it is argued above, the current can not reverse the sign of its component along the tangent vector of line of forces. Therefore, it follows that also the electric field can‘t revert the sign of its component along a line of force. Thus, the integrand of the line integral over the C_1 curve should have a definite sign at all the points, hence implying that ϕ and the electric field should vanish exactly in all C_1. Resuming, it follows that the electric field vanish also at the interior of any of the closed surfaces S_i. Therefore, the conclusion arises that the difference solution $\phi = \phi_1$-$\phi_2 = 0$ in all the space, thus showing that the evoked potential at B_0 uniquely fixes the sources when they have their support in a set of non nesting surfaces S.

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Figure Captions

Fig.1. An illustration of a simply connected region R constituted in this case by only two simply connected subregions R_1 and R_2 having a boundary B_{12}.The boundary with free space is denoted by B_0. The set of non-nesting surfaces S have four elements S_i , $i = 1, ..4$. two of them open and other two closed ones. A piece wise straight curve C joining any interior point P of R and a point O in the free space is also shown.

Fig.2. Picture representing the region Q in which a field φ satisfies the Laplace equation and its value at the point $\overrightarrow{x}$ is given by the Green integral (5).

Fig.3. The contour employed in the line integral in Eq. (6).

Fig.4. Picture of the region R_i and the open neighborhood N in which the field φ vanish exactly . A piece wise straight line curve C joining a point $P \in N$ and certain point P_1 in R_i is also shown.

Fig.5. Scheme of the curve C and the open region O_C containing it.

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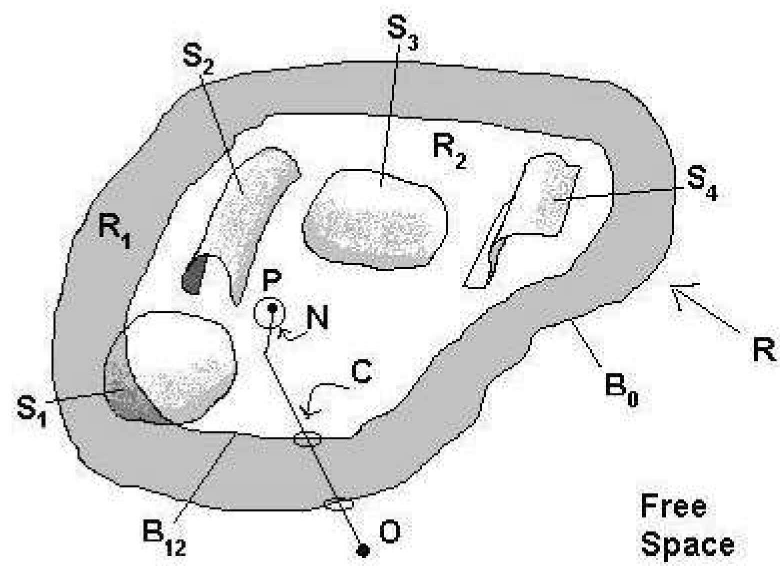

Fig. 1

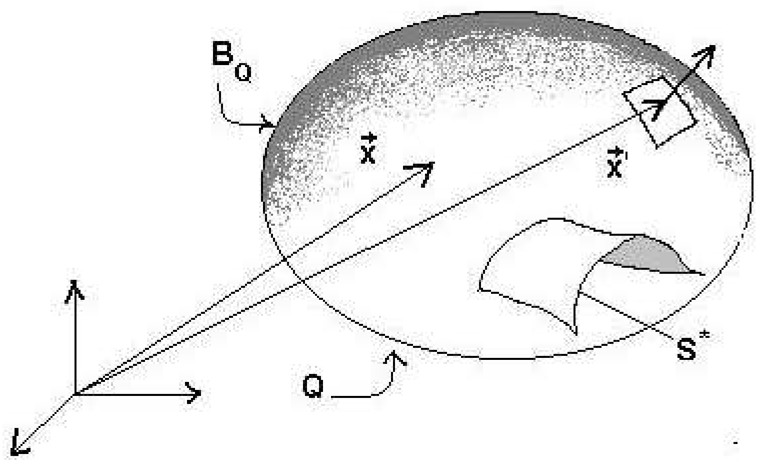

Fig. 2

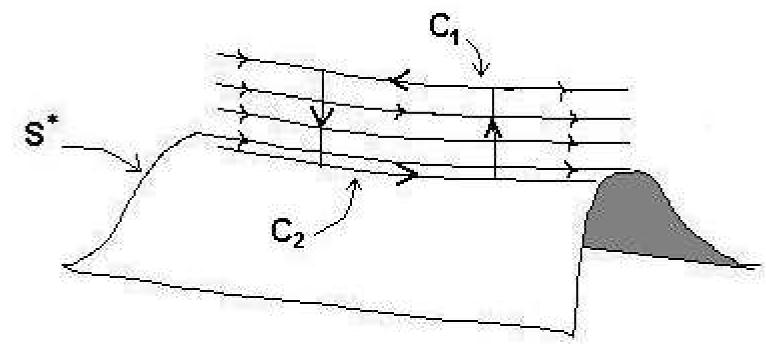

Fig. 3

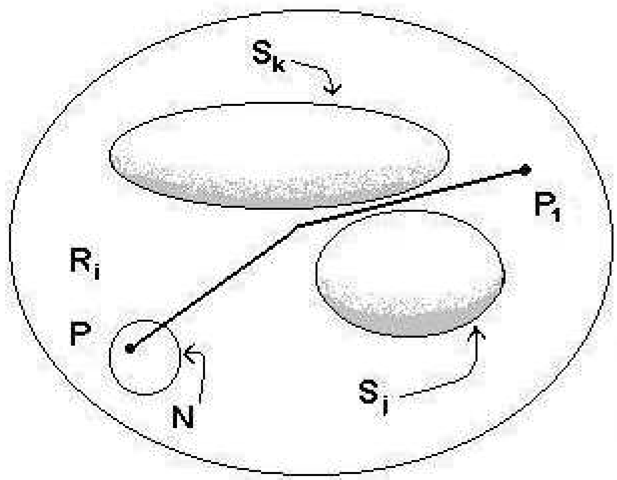

Fig. 4

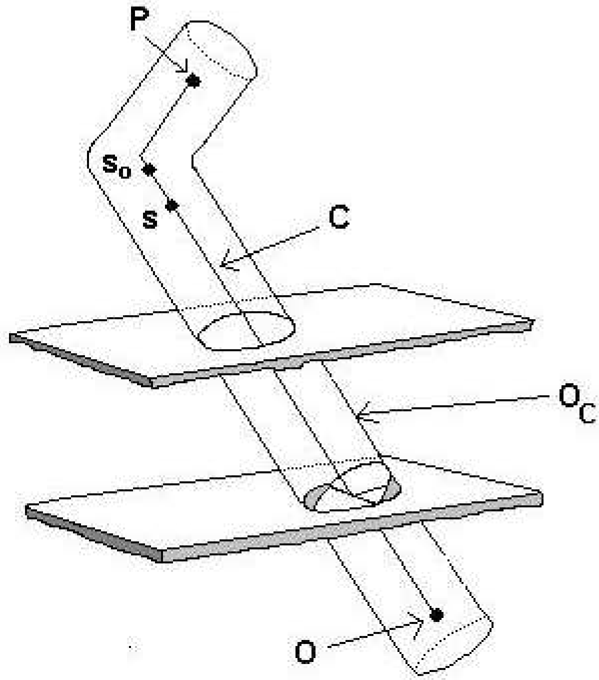

Fig. 5